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1981 J. Phys. A: Math. Gen. 14 745

(http://iopscience.iop.org/0305-4470/14/3/021)

J. Phys. A: Math. Gen. 14 (1981) 745–751. Printed in Great Britain

Some remarkable spin-$\frac{1}{2}$-like algebraic properties of spin-$\frac{3}{2}$ matrices

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Received 27 June 1980

Abstract. Using for spin-$\frac{3}{2}$ matrices a direct-product structure involving the usual Pauli spin matrices, we derive the Dirac–Clifford matrices in terms of certain algebraic combinations of spin-$\frac{3}{2}$ matrices in a representation-independent way, thus achieving a newer extension of the Pauli spin matrices from the usual spin-$\frac{1}{2}$ space to the spin-$\frac{3}{2}$ space. Basing the derivation directly on this analysis, we derive two algebras satisfied by spin-$\frac{3}{2}$ matrices. We demonstrate that one of these which is also satisfied by spin-$\frac{1}{2}$ matrices is directly related to the spin-$\frac{3}{2}$ algebras of Weaver and of Bhabha and Madhava Rao (for three objects). The other algebra is new and curiously is not satisfied by spin-$\frac{1}{2}$ matrices.

1. Introduction

In a recent paper Weaver (1978a) has projected the importance of spin-$\frac{1}{2}$-like properties of certain algebraic combinations of spin-1 matrices for tackling problems involving aspects of spin-1 matrix algebras. Though Weaver (1978b) has subsequently considered an algebra for spin-$\frac{3}{2}$ matrices, the analogous spin-$\frac{1}{2}$-like properties for the spin-$\frac{3}{2}$ case do not seem to have been reported in the literature. In this paper we address ourselves to this task and derive, by effectively exploiting the existence of a direct-product structure for spin-$\frac{3}{2}$ matrices, the Dirac–Clifford matrices in terms of certain algebraic combinations of spin-$\frac{3}{2}$ matrices. Based directly on this correspondence, which we also demonstrate to be valid for any arbitrary representation of spin-$\frac{3}{2}$ matrices, we derive two algebras for spin-$\frac{3}{2}$. We point out that one of these algebras, which is also satisfied by spin-$\frac{1}{2}$ matrices, is directly related to the algebra of Weaver (1978b) and in form to the algebra employed earlier by Bhabha and Madhava Rao (Corson 1953) in connection with Bhabha first-order wave equations (Krajcik and Nieto 1975) for maximum spin-$\frac{3}{2}$, as restricted to the case of three objects. We also present explicitly the other newer algebra which is not satisfied by spin-$\frac{1}{2}$ matrices but only by spin-$\frac{3}{2}$ matrices. While Weaver, for the derivation of his algebra, uses the Lorentz transformation properties of the symmetric, traceless, covariantly defined spin tensor (Barut *et al* 1963, Weinberg 1964) related to the spin matrices, our method in a transparent way projects the spin-$\frac{1}{2}$-like properties of spin-$\frac{3}{2}$ matrices from the beginning and employs the aid of no more than the basic angular momentum commutation relations of the spin-$\frac{3}{2}$ matrices, their characteristic equations and the existence of the direct-product structure aforementioned. Our method also yields a newer algebra.

0305-4470/81/030745+07$01.50

2. Dirac–Clifford matrices and spin-$\frac{3}{2}$ matrices

We shall now proceed to prove our assertions. We work with the system of units $c = \hbar = 1$. We start with the spin-$\frac{3}{2}$ matrices S_i $(i = 1, 2, 3)$ satisfying the following basic properties:

$$[S_i, S_j]_- = \mathrm{i}\varepsilon_{ijk}S_k, \tag{1a}$$

$$S_i^4 = \tfrac{5}{2}S_i^2 - \tfrac{9}{16}, \tag{1b}$$

$$\sum_{i=1}^{3} S_i^2 = \tfrac{15}{4}. \tag{1c}$$

Now expressing S_i in a direct-product form

$$S_i = s_i \otimes (\sigma_i/2) \tag{2}$$

where σ_i are the two-dimensional Pauli spin matrices satisfying

$$\sigma_i\sigma_j = \delta_{ij} + \mathrm{i}\varepsilon_{ijk}\sigma_k, \tag{3}$$

equations (1) lead, by virtue of (3), to the following requirements on s_i which, without loss of generality, are

$$[s_i, s_j]_+ = 2s_k, \tag{4a}$$

$$s_i^2 = -2s_i + 3, \tag{4b}$$

$$\sum_{i=1}^{3} s_i = -3. \tag{4c}$$

That there always exist such Hermitian s_i satisfying (4) corresponding to an appropriately unitary-transformed arbitrary Hermitian representation of spin-$\frac{3}{2}$ matrices can be ascertained by considering the following example, where the unitary matrix

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

satisfies

$$UJ_iU^\dagger = S_i = s_i \otimes (\sigma_i/2) \qquad (i = 1, 2, 3) \tag{5a}$$

with J_i being the spin-$\frac{3}{2}$ matrices in the usual representation (Schiff 1968) and

$$s_1 = \begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & -2 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & -2 \end{pmatrix}, \qquad s_3 = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5b}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5c}$$

However, for our analysis we will employ only the algebraic properties (3) and (4) of σ_i and s_i respectively and not the explicit matrix realisations (5*b*, *c*).

Employing (4*c*), we have

$$\left[s_i, \sum_{l=1}^{3} s_l\right]_- = 0 \qquad (i = 1, 2, 3) \tag{6}$$

which immediately leads to

$$[s_1, s_2]_- = [s_2, s_3]_- = [s_3, s_1]_- = \mu \qquad \text{(say)}. \tag{7}$$

Feeding (7) into (4*a*), we also obtain

$$s_i s_j = s_k + \tfrac{1}{2}\varepsilon_{ijk}\mu_k, \qquad \mu_k = \mu \qquad (i \neq j \neq k \neq i). \tag{8}$$

With (4*b*), (7) and (8), we easily derive that

$$\mu = \tfrac{1}{3}(s_j - s_i)(2s_k - s_i - s_j) \qquad (i \neq j \neq k \neq i, \quad i, j, k \text{ cyclic}) \tag{9}$$

and equations (4) further furnish

$$\left.\begin{aligned} &[(1/2\sqrt{3})(s_j - s_i)]^2 = 1 && (10a)\\ &[\tfrac{1}{6}(2s_k - s_i - s_j)]^2 = 1 && (10b)\\ &\tfrac{1}{12}[s_i - s_j, s_i - s_k]_+ = 1 && (10c)\\ &[s_i - s_j, 2s_k - s_i - s_j]_+ = 0 && (10d)\end{aligned}\right\} \quad (i \neq j \neq k \neq i).$$

By combining the information of (4*c*) and (7) it is not difficult to derive

$$[s_i, \mu]_+ = [s_j, \mu]_+ = -2\mu \qquad (i \neq j) \tag{11a}$$

which leads immediately to

$$[\mu, s_j - s_i]_+ = [\mu, 2s_k - s_i - s_j]_+ = 0 \qquad (i \neq j \neq k \neq i). \tag{11b}$$

Equations (9)–(11) imply that

$$(\mu/4\sqrt{3})^2 = -1. \tag{12}$$

Now equations (1*a*), (2), (3) and (8) straightaway lead to

$$\alpha_k = (2/\sqrt{3})(S_i S_j - \tfrac{1}{2}\mathrm{i}S_k) \qquad (i \neq j \neq k \neq i, \quad i, j, k \text{ cyclic}) \tag{13a}$$

or equally well to

$$\alpha_k = (1/\sqrt{3})[S_i, S_j]_+ = (\mathrm{i}/4\sqrt{3})\mu \otimes \sigma_k \qquad (i \neq j \neq k \neq i), \tag{13b}$$

satisfying, by virtue of (3) and (12),

$$[\alpha_k, \alpha_l]_+ = 2\delta_{kl} \qquad (k, l = 1, 2, 3) \tag{13c}$$

thus providing the first three anticommuting Dirac–Clifford matrices. The remaining two Dirac–Clifford matrices are also easily deduced by the observation that

$$\alpha_4 = \frac{1}{\sqrt{3}}(S_i^2 - S_j^2) = \frac{1}{2\sqrt{3}}(s_j - s_i) \otimes I \qquad (i \neq j), \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{14a}$$

and

$$\alpha_5 = \tfrac{1}{3}(S_i^2 + S_j^2 - 2S_k^2) = \tfrac{1}{6}(2s_k - s_i - s_j) \otimes I \qquad (i \neq j \neq k \neq i) \tag{14b}$$

satisfy, by virtue of (10),

$$\alpha_4^2 = \alpha_5^2 = 1, \tag{14c}$$

$$[\alpha_4, \alpha_5]_+ = 0 \tag{14d}$$

and also, by virtue of (11*b*) and (13*b*),

$$[\alpha_4, \alpha_i]_+ = [\alpha_5, \alpha_i] = 0 \qquad (i = 1, 2, 3). \tag{14e}$$

Equations (13) and (14) complete our assertion of construction of the Dirac–Clifford matrices α_ρ satisfying

$$[\alpha_\rho, \alpha_\nu]_+ = 2\delta_{\rho\nu} \qquad (\rho, \nu = 1, 2, 3, 4, 5). \tag{15}$$

It may be noted that (14*a*, *b*) provide only two independent mutually anticommuting matrices because of (10). Though we have so far worked by exploiting the direct-product structure (2) of spin-$\frac{3}{2}$ matrices, the algebraic expressions and properties (13)–(15) for the α_ρ matrices are valid in terms of any arbitrary representation (not necessarily the direct-product) for spin-$\frac{3}{2}$ because of the connecting unitary transformation between an arbitrary representation and the direct-product form as exemplified in equations (5).

In fact, with the use of the explicit matrices s_i and σ_i of (5) in (2), we have the following explicit expressions for the α_μ:

$$\begin{aligned}
\alpha_1 &= (1/\sqrt{3})[S_2, S_3]_+ = -\sigma_2 \otimes \sigma_1 = \mathrm{i}\mathrm{IV}_{34},\\
\alpha_2 &= (1/\sqrt{3})[S_3, S_1]_+ = -\sigma_2 \otimes \sigma_2 = \mathrm{IV}_{44},\\
\alpha_3 &= (1/\sqrt{3})[S_1, S_2]_+ = -\sigma_2 \otimes \sigma_3 = \mathrm{i}\mathrm{IV}_{43},\\
\alpha_4 &= (1/\sqrt{3})(S_1^2 - S_2^2) = \sigma_1 \otimes I = \mathrm{IV}_{13},\\
\alpha_5 &= \tfrac{1}{3}(S_1^2 + S_2^2 - 2S_3^2) = -\sigma_3 \otimes I = -\mathrm{IV}_{31}.
\end{aligned} \tag{16}$$

In (16), IV_{ij} $(i, j = 1, 2, 3, 4)$ are five of the sixteen coordinate interchange matrices recently considered by Stephany (1979) in connection with a four-dimensional extension of the Pauli spin matrices. The remaining IV_{ij} of Stephany can be directly written out as bilinear products amongst α_μ of (16) but is not presented here.

It is remarkable at this point to observe that the Dirac matrices of equation (15), with their basic definitions as provided by equations (13*b*) and (14*a*, *b*) in terms of spin-$\frac{3}{2}$ matrices, indeed constitute a new representation-independent extension of the usual Pauli spin matrices of equation (3) from the spin-$\frac{1}{2}$ space to the spin-$\frac{3}{2}$ space.

3. Spin-$\frac{3}{2}$ algebras

Now we shall show that two algebras emerge for the S_i's based directly on the α_ρ algebra (15), as given explicitly in terms of the S_i's. While relations (13) take the form

$$[[S_i, S_j]_+, [S_k, S_l]_+]_+ = 6\delta_{ik}\delta_{jl} + 6\delta_{il}\delta_{jk} \qquad (i \neq j,\ k \neq l), \tag{17a}$$

the first of relations (14*e*) can be recast in the equivalent form

$$\begin{aligned}
[[S_i, S_i]_+, [S_l, S_m]_+]_+ &= [[S_j, S_j]_+, [S_l, S_m]_+]_+ = [[S_k, S_k]_+, [S_l, S_m]_+]_+\\
&= 5[S_l, S_m]_+ \qquad (i \neq j \neq k \neq i,\ l \neq m),
\end{aligned} \tag{17b}$$

in deriving which use has been made of (1*c*). We also observe that while the characteristic equation (1*b*) takes the form

$$[[S_i, S_i]_+, [S_i, S_i]_+]_+ = 20S_i^2 - \tfrac{9}{2}, \tag{17c}$$

the relation (14*d*) together with (1*b*, *c*) yields

$$[[S_i, S_i]_+, [S_j, S_j]_+]_+ = 10(S_i^2 + S_j^2) - \tfrac{33}{2} \qquad (i \neq j). \tag{17d}$$

It is straightforward, though a little tedious, to subsume the content of equations (17) into either of the following single forms:

$$[[S_i, S_j]_+, [S_k, S_l]_+]_+ = -\tfrac{3}{2}(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk}+\delta_{lj}\delta_{ki})+3\delta_{kl}[S_i, S_j]_+ + 3\delta_{ij}[S_k, S_l]_+ + \delta_{ik}[S_j, S_l]_+ + \delta_{il}[S_j, S_k]_+ + \delta_{jk}[S_i, S_l]_+ + \delta_{jl}[S_i, S_k]_+ + (\varepsilon_{kim}\varepsilon_{jln}+\varepsilon_{lim}\varepsilon_{jkn})[S_m, S_n]_+, \tag{18}$$

$$[[S_i, S_j]_+, [S_k, S_l]_+]_+ = -\tfrac{33}{2}\delta_{ij}\delta_{kl}+6\delta_{ik}\delta_{jl}+6\delta_{il}\delta_{jk}+5\delta_{ij}[S_k, S_l]_+ + 5\delta_{kl}[S_i, S_j]_+. \tag{19}$$

In other words, equations (17) comprise the statement of algebraic properties (18) or (19) for spin-$\frac{3}{2}$. In fact, on expansion, the RHS of (18) is identically equal to the RHS of (19) only for spin-$\frac{3}{2}$. However, viewed as an algebra, (18) is satisfied also by spin-$\frac{1}{2}$ matrices, i.e. with the substitution of $S_i = \sigma_i/2$, but (19) is not. Incidentally, algebra (19) can also be rewritten in a more elegant form:

$$[[S_i, S_j]_+ - \tfrac{5}{2}\delta_{ij}, [S_k, S_l]_+ - \tfrac{5}{2}\delta_{kl}]_+ = 6\delta_{ik}\delta_{jl}+6\delta_{il}\delta_{jk}-4\delta_{ij}\delta_{kl}.$$

Now algebra (18) together with (1a) is equivalent to the Bhabha–Madhava Rao-like algebra (Corson 1953) as specialised for three objects,

$$[S_i, S_jS_kS_l+S_lS_kS_j]_+ = -\tfrac{3}{4}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{lj}+\delta_{il}\delta_{jk})+\tfrac{3}{2}\delta_{kl}[S_i, S_j]_+ + \tfrac{3}{2}\delta_{jk}[S_i, S_l]_+ + \tfrac{1}{2}\delta_{ij}[S_k, S_l]_+ + \tfrac{1}{2}\delta_{ik}[S_j, S_l]_+ + \tfrac{1}{2}\delta_{il}[S_k, S_j]_+ + \tfrac{1}{2}\delta_{jl}[S_i, S_k]_+, \tag{20}$$

which is satisfied by both spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ matrices. The above equivalence can be demonstrated after some labour by first deducing the information of a single term $S_iS_jS_kS_l$ obtained from (18) with repeated use of (1a) and then summing up for the RHS of (20) to arrive finally at the algebra (20). A similar treatment carried out for (19), however, yields a new algebra:

$$[S_i, S_jS_kS_l+S_lS_kS_j]_+ = \tfrac{3}{16}\delta_{ij}\delta_{lk}+\tfrac{3}{16}\delta_{il}\delta_{jk}-\tfrac{21}{8}\delta_{ik}\delta_{jl}+\tfrac{11}{8}\delta_{kl}[S_i, S_j]_+ + \tfrac{11}{8}\delta_{jk}[S_i, S_l]_+ + \tfrac{3}{4}\delta_{ik}[S_j, S_l] + \tfrac{3}{4}\delta_{lj}[S_i, S_k]_+ + \tfrac{3}{8}\delta_{il}[S_k, S_j]_+ + \tfrac{3}{8}\delta_{ij}[S_k, S_l]_+ + \tfrac{1}{8}(\varepsilon_{lim}\varepsilon_{jkn}+\varepsilon_{ijm}\varepsilon_{kln})[S_m, S_n]_+ \tag{21}$$

which is satisfied by spin-$\frac{3}{2}$ matrices but, curiously enough, not by spin-$\frac{1}{2}$ matrices.

The following algebra of Weaver (1978b), involving twelve terms of the type $S_iS_jS_kS_l$,

$$\left[S_i, \frac{1}{3!}(S_jS_kS_l+S_lS_kS_j)+\frac{1}{3!}(S_kS_lS_j+S_jS_lS_k)+\frac{1}{3!}(S_kS_jS_l+S_lS_jS_k)\right]_+ = -\tfrac{3}{8}(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl})+\tfrac{1}{4}(\delta_{il}[S_j, S_k]_+ + \delta_{ik}[S_j, S_l]_+ + \delta_{ij}[S_k, S_l]_+) + \tfrac{7}{12}(\delta_{kl}[S_i, S_j]_+ + \delta_{jk}[S_i, S_l]_+ + \delta_{jl}[S_i, S_k]_+), \tag{22}$$

is also satisfied by spin-$\frac{1}{2}$ matrices, and together with (1a) is indeed equivalent to the Bhabha–Madhava Rao-like algebra (20), involving only four terms of the type $S_iS_jS_kS_l$

on the LHS. This equivalence can be established in a straightforward fashion by the use of ($1a$) first to deduce

$$[S_j, [S_k, S_l]_-]_- = \varepsilon_{klm}\varepsilon_{jnm}S_n = \delta_{jk}S_l - \delta_{jl}S_k, \tag{23}$$

then the use of (23) to obtain

$$S_kS_lS_j + S_jS_lS_k = S_jS_kS_l + S_lS_kS_j + \delta_{lj}S_k - \delta_{kj}S_l \tag{24a}$$

and

$$S_lS_jS_k + S_kS_jS_l = S_jS_kS_l + S_lS_kS_j + \delta_{lj}S_k - \delta_{lk}S_j, \tag{24b}$$

and the subsequent use of (24) on the LHS of (22) to eliminate the terms in the second and the third parentheses within the anticommutator, to result finally in the algebra (20).

The algebra (21) in its content for spin-$\frac{3}{2}$ is however consistent with Weaver's algebra (22), as can be verified directly starting from the LHS of (22), rendering this into three anticommutators each involving terms contained in the respective three parentheses, and then substituting for each of them the information as contained in the algebra (21). The reason that Weaver's algebra is also satisfied by spin-$\frac{1}{2}$ matrices is not far to seek, as in fact the defining algebraic relations (equation (9) of Weaver (1978b)) employing the Lorentz transformation properties of spin tensors as specialised for spin-$\frac{3}{2}$, which Weaver has used to deduce his algebra (22), are also satisfied by the simple substitution of $S_i = \frac{1}{2}\sigma_i$.

Weaver has used the algebra (22) together with ($1a$) and the commutation relations of the components of $\pi_i = p_i - eA_i$ to deduce a characteristic equation for $\boldsymbol{S} \cdot \boldsymbol{\pi}$ and thus the eigenvalues of $\boldsymbol{S} \cdot \boldsymbol{\pi}$ for spin-$\frac{3}{2}$. Since the algebra (22) of Weaver and the basic angular momentum commutation relations ($1a$) are also satisfied by spin-$\frac{1}{2}$ matrices, one expects that the characteristic equation for $\boldsymbol{S} \cdot \boldsymbol{\pi}$, as deduced by Weaver (1978b) based on algebra (22), will also be satisfied by $\frac{1}{2}(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})$. Hence one surmises that the use of the new algebra (21) that is not satisfied by spin-$\frac{1}{2}$ matrices may lead to a different characteristic equation for $\boldsymbol{S} \cdot \boldsymbol{\pi}$ not satisfied by $\frac{1}{2}(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})$. The details of our current calculations on this point will however be reported in a future communication. Also, the spin-$\frac{1}{2}$-like properties of the algebraic combinations of spin-$\frac{3}{2}$ matrices, as exemplified by equations (13)–(15), coupled with the aforementioned observation that $\frac{1}{2}(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})$ will satisfy the same characteristic equation as $\boldsymbol{S} \cdot \boldsymbol{\pi}$ of the spin-$\frac{3}{2}$ case based on the algebra (20), have a crucial role to play in resolving an apparent paradox recently pointed out by Weaver (1977) on the predictive power, in the context of an external electromagnetic interaction, of the new linear Dirac-like wave equation for spin-$\frac{3}{2}$ derived recently by one of the authors (Jayaraman 1976). A detailed discussion of this will be the subject matter for a separate publication.

Acknowledgments

The authors wish to express their thanks to Professors Nelson Lima Teixeira, Waldyr Alves Rodrigues Jr, João Goedert and Júlio de Melo Teixeira of the Federal University of Paraíba, Brazil for their kind hospitality and constant encouragement. One of us (FEAS) is grateful to the Federal University of Acre, Brazil for the grant of a study leave.

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